Chapter 8

Conclusion

This work investigated the applicability of semiclassical approximations to mesoscopic systems. The different problems analyzed are grouped around three setups: two simple model systems and a more complicated structure realized in experiment. The theoretical studies, namely the calculation of the level density of the disk billiard and the conductivity tensor of the free 2DEG, analyzed the influence of various higher-order $\hbar$ contributions to the semiclassical description. It was shown that only a few of these corrections are relevant. The magnetoconductance of the experimental system – a mesoscopic channel with antidots – was successfully described within leading order of $\hbar$. All observed features were quantitatively and qualitatively reproduced, and an intuitive picture of their origin could be given.

Contents

Working on semiclassical techniques, i.e. approximations of the quantum mechanical formalism to leading order in $\hbar$, can have different foci. The interest can be directed towards the correct description of the properties of a specific system. This includes all questions relevant for (experimental) applications. Another motivation is to gain increased theoretical insight in semiclassical techniques. In this context, the limits of such an approximation are considered and formal problems of the approach are solved.

For the present work, these two aspects have been equally important. This conclusion will therefore summarize the main results twice. First, an overview of the findings for the specific systems is given (Sec. 8.1), before the results are collected according to theoretical criteria (Sec. 8.2 and 8.3). The dissertation closes with suggestions for further investigations.

8.1 The systems investigated

The semiclassical approximation was shown to be a well-adapted tool for all the mesoscopic systems considered. The approach was seen to be sufficiently accurate. Compared to the quantum mechanical calculations, the numerical effort is significantly reduced.
The semiclassical trace formula only depends of the properties of the classical dynamics of the system. This feature allows to give simple, intuitive pictures for the relevant processes. A particular focus was directed towards the explanation of the observed quantum interference effects in these classical terms.

8.1.1 The disk billiard

The level density of the disk billiard in homogeneous magnetic field was chosen as the first model system. It is simple enough to allow an analytic quantum mechanical solution, so that the shortcomings of the semiclassical approximation can be analyzed very accurately. It is, on the other hand, complex enough to exhibit the typical problems semiclassical approximations face. In varying magnetic fields, orbit bifurcations occur. Additionally, the system comprises orbits with symmetries of different dimensionality, so that different powers in $\hbar$ contribute to the trace sum. Finally, grazing or diffractive effects come into play. A detailed analysis resulted in a surprisingly small influence of most of these $\hbar$ corrections. The only relevant correction concerns the Maslov index. Using a simple one-dimensional approximation, it was replaced by a reflection phase. With this modification, the semiclassical trace formula gives excellent results for arbitrary field strengths, both for the shell structure and for full quantization. The shell structure could be explained within a simple picture including only the classical properties of three periodic orbits.

8.1.2 The free 2DEG

The free 2DEG was selected as a simple model system for the application of semiclassical transport theory. Both its longitudinal and its Hall conductivity were described semiclassically. It was shown that the Shubnikov-de-Haas oscillations in the longitudinal conductivity are an effect of leading order in $\hbar$. The integer quantum hall effect, in contrast, is almost exclusively due to a single contribution of *second* leading order, namely the period dependence of the cyclotron orbits on the magnetic field strength. The corresponding $\hbar$ correction could be derived for arbitrary systems.

8.1.3 The channel with antidots

The longitudinal magnetoconductance of a narrow channel with central antidots was chosen deliberately as a difficult problem for a semiclassical analysis. Previous results had found strong indications that the effects observed in this system, namely the dependence of the maxima spacings on the magnetic field and the dislocations observed when varying the antidot diameter, are genuine quantum effects, not accessible to a semiclassical calculation. Furthermore, the system shows features that disfavor a semiclassical description: The channel has many orbits with comparable length and action. They are related by a complicated structure of bifurcations, and ghost orbits are ubiquitous. Despite these problems and the related technical difficulties it was possible to give a complete semiclassical description of all observed effects. The local behavior at a dislocation could even be reproduced quantitatively, and the parameters of the model potential could be fit to the experimental situation. It was shown that the complex behavior of the system is not due to higher-order contributions in $\hbar$, but to a delicate interplay between the actions and amplitudes of many similar orbits. Within the semiclassical picture, the observed magnetoconductance features could be related to bifurcations in the leading classes of periodic orbits.

8.2 The relevance of $\hbar$ corrections

A central result of this work is that semiclassical approximations are valid in a much wider range than they are expected to. Even large individual higher-order contributions frequently lead to marginal corrections, since they average out to a great extent. Only two important sources of $\hbar$ corrections were identified.

8.2.1 Different powers in $\hbar$ in the trace formula

The first group of relevant $\hbar$ corrections are those that come naturally into play by the application of the semiclassical procedure itself. This happens for example in systems that include orbits symmetries with different dimensionality. The orbits with lower symmetry lead to contributions in higher than leading order in $\hbar$. These $\hbar$ corrections are neglected in the standard approach. For the disk billiard, however, the inclusion of the lower-symmetry bouncing orbits was shown to be necessary.[1]

The conductivity tensor is another property where the semiclassical approximation by itself includes higher-order terms in $\hbar$. In this case the leading-order term of the level density shows up as an $\hbar$ correction in the Hall conductivity. This correction is responsible for the plateaus in the hall voltage, i. e. the integer quantum hall effect. The corresponding formula for general systems could be derived.

8.2.2 Reflection phases

The Maslov index was identified as another source of potentially important $\hbar$ corrections. This additional phase occurs at classical turning points (or their higher-dimensional equivalents). It exhibits a spurious discontinuity in dependence of the steepness of the

[1] Similar results have been obtained for other systems as well [119].

effective potential. In a one-dimensional approximation this jump can be removed by replacing the Maslov index by a reflection phase. The phase depends explicitly on $\hbar$, which corresponds to $\hbar$ corrections to the trace formula. These corrections were shown to be important for the disk billiard in the strong field regime $R_c \lesssim R$. They correct the degeneracy of the Landau levels, which is overestimated by the standard approach. The mechanism can be interpreted as a proximity effect of the boundary.[2]

The generalization of the reflection phase to higher dimensions is still an open problem.

8.2.3 Bifurcations and grazing

Grazing corrections are related to finite integration limits in stationary-phase approximations. These effects can be expected to give corrections up to 50% to the level density of the disk billiard. A detailed analysis showed, however, that this correction is negligible there.

Bifurcations were seen to be important for problems including only a few orbits. This is intuitive, since bifurcations lead to divergencies in the standard Gutzwiller approach. If many orbits are included, the influence of the bifurcations (although their number might drastically increase) gets smaller. This has to be interpreted as a cancellation effect. This behavior was found for both the disk billiard and the channel system.

There are different mechanism relating bifurcations to experimentally observable effects. It was reported that their lower order in $\hbar$ may lead to local dominance of the bifurcations. Period doubling bifurcations are known to be responsible for period-doubling effects in resonant tunneling diodes. The mechanism in the channel was shown to be different. There, bifurcations mark the boundaries of phase space regions with different classical dynamics. The semiclassical description reflects the classical phase space structure in the quantum oscillation. Like that is was possible to directly relate one type of bifurcations of the cannel system to the dislocations observed in the maxima positions of the magnetoconductance.

8.3 Smoothing in higher order of $\hbar$

Finite temperature and impurity scattering lead to finite line widths and characteristic line shapes. Just as the semiclassical approximation itself, the implementation of those effects in the trace formula is only valid up to leading order in $\hbar$. This work presented generalizations applicable to the $\hbar$ corrections considered.

For both cases where this modified smoothing scheme was applied, namely the bifurcations and the grazing correction in the disk billiard, the magnitude of the smoothing correction was comparable to the $\hbar$ correction itself. For bifurcations in systems with many orbits it was shown that neglecting the $\hbar$ correction in the trace formula and just including the correct (numerical) smoothing already leads to surprisingly good results.

[2] Reflection phases can also be applied to problems including partial transmission and reflection, and therefore also to tunneling.

8.4 Suggestions for further investigations

The results of this work indicate that bifurcations have a considerable influence in systems with only a few dominating orbits. For those, a numerical inclusion of bifurcations of codimension 2 along the path presented here could be useful. A suitable prototype for these systems is the Hénon-Heiles potential, where three important orbits are involved in a period-tripling and subsequent tangent bifurcation. An appropriate starting point are the uniform formulas presented by Schomerus [71], with an adaption to the numerical limitations following appendix B.

For the disk billiard in strong magnetic fields, the correction of the Maslov index was shown to be important. This correction was included using a simple, one dimensional approximation, where the potential at the classical turning point was expanded to linear order. This approximation could be further improved implementing the analytical reflection phases worked out by Friedrich *et al.* [29]. This would be another step towards a generalization of the Maslov index.

The most rewarding field for future investigations, however, is the application of the semiclassical techniques to experiment. The fit of the model potential of the channel system was so successful, that a project to determine the experimental potential by semiclassical techniques seems reasonable. Such a project would exploit the substantially reduced numerical effort of semiclassical calculations, making an adaption of many parameters of the experimental potential numerically feasible. A good candidate for such an investigation is the conductance of the Aharonov-Bohm-ring measured by Pedersen *et al.* [124]. The long-range oscillations superimposed on the AB frequency stem from the interference of trapped orbits in the ring arms. Fitting the semiclassical findings to the experimental observations should, among other predictions, allow an estimate of the depletion width of the etch border.

Similar calculations should be possible for antidot superlattices. For the system with large antidot diameter presented by Eroms *et al.* [27], also the steepness of the effective potential could be accessible.

These investigations would add a new quality to semiclassical approximations. The semiclassical description would then not only reproduce the experimental findings and allow their interpretation in an intuitive picture. For these systems, it could additionally provide a tool to determine experimental parameters hardly accessible by other means.